A Distributed Community Detection Algorithm for Large Scale Networks Under Stochastic Block Models

Zhe Li

Joint work with Shihao Wu and Xuening Zhu

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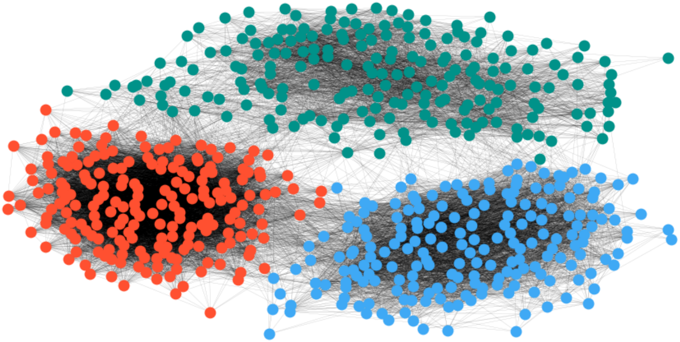
Outline

Introduction

- Communities in Yelp dataset (https://www.yelp.com/dataset)



Introduction

- Community detection is a fundamental task within network analysis
- Numerous methodologies exist for this task.:
  - Likelihood based methods (Zhao et al. 2012)
  - Convex Optimization (Chen et al., 2012)
  - Methods of moments (Anandkumar et al., 2014)
  - Spectral clustering (Rohe et al., 2011; Lei and Rinaldo, 2015);

Algorithm 1: Spectral Clustering for SBM (SC)

Input: Adjacency matrix A; number of communities K.

Output: Membership matrix $\widehat{\Theta}$.

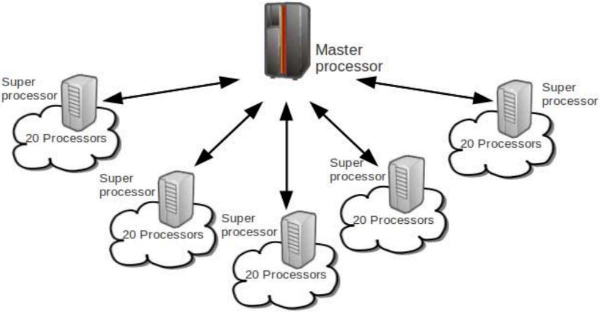
1: Compute Laplacian matrix L based on A.

2: Conduct eigen-decomposition of L and extract the top K eigenvectors (i.e., $\widehat{U}$).

3: Conduct k-means algorithm using $\widehat{U}$ and then output the estimated membership matrix $\widehat{\Theta}$.

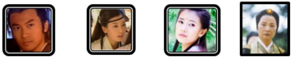
Introduction

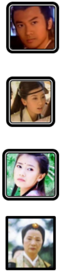
- What if the network is of large scale? $\Rightarrow$ great computational power
- privacy? $\Rightarrow$ stored in a distributed manner across various data centers.



Can we consider a distributed algorithm for the spectral clustering?

Distributed Community Detection under SBM





	1	1	1
1		1	
1	1		1
1		1	

- Adjacency matrix $A = (a_{ij})$
- $a_{ij} = 1$ indicates the ith user follows the jth user; otherwise $a_{ij} = 0$.

Stochastic Block Model: Membership matrix

- $\Theta = (\Theta_1, \cdots, \Theta_N)^\top \in \mathbb{R}^{N \times K}$
- For the ith row of Θ, only the g_ith element takes 1 and the others are 0.
- The membership matrix of the left figure is:

$$\begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

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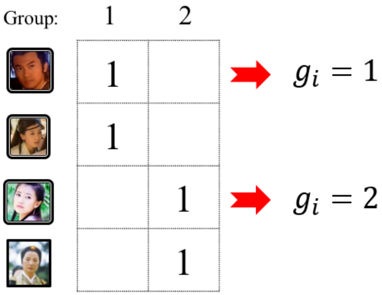
Stochastic Block Model: Connectivity Matrix



- $B \in \mathbb{R}^{K \times K}$ with full rank
- The connection probability between the kth and lth community is B_{kl}
- The element A_{ij} in the adjacency matrix is generated independently from Bernoulli($B_{g_i g_j}$) distribution.

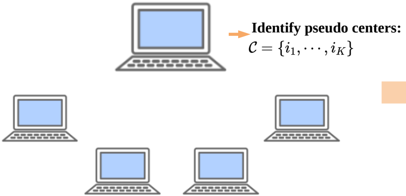
Spectral Clustering under SBM

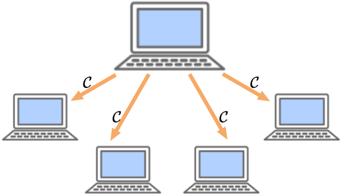
Lemma 1. (Lemma 3.1 in Rohe et al. (2011)).

The eigen-decomposition of $\mathcal{L}$ takes the form $\mathcal{L} = U\Sigma U^\top$, where $U = (U_1, \cdots, U_N)^\top \in \mathbb{R}^{N\times K}$ collects the eigen-vectors and $\Sigma \in \mathbb{R}^{K\times K}$ is a diagonal matrix. Further we have $U = \Theta\mu$, where μ is a $K \times K$ orthogonal matrix and $\Theta_i = \Theta_j$ if and only if $U_i = U_j$.

- $\mathcal{L} = \mathcal{D}^{-1/2}\mathcal{A}\mathcal{D}^{-1/2}$, where $\mathcal{A} = \mathbb{E}(A)$ and $\mathcal{D} = \mathbb{E}(D)$
- U only has K distinct rows and the ith row is equal to the jth row if the corresponding two nodes belong to the same community

A Distributed Algorithm





Step 1:

- Conduct spectral clustering on master server to identify pseudo centers.

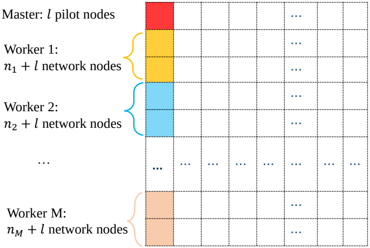
Step 2:

- Broadcast pseudo centers to workers
- Complete distributed community detection task using a SVD type algorithm.

Pilot Network Spectral Clustering on Master Server

- Suppose we have l network nodes on the master $\Rightarrow$ **pilot nodes**.
- In addition we distribute the pilot nodes both on master and workers.

- Conduct the spectral clustering on the pilot network $A_0 \in \mathbb{R}^{l \times l}$ and obtain the clustering centers $\widehat{C}_0 = \left(\widehat{C}_{0k} : 1 \le k \le K\right)^\top$.
- Determine the indexes of the kth pseudo centers as $i_k = \arg\min_i \left\|\widehat{U}_{0i} - \widehat{C}_{0k}\right\|_2^2$.
- Broadcast the index set of pseudo centers $\mathcal{C} = \{i_1, \cdots, i_K\}$ to workers.



Community Detection on Workers

- Suppose we distribute n_m network nodes as well as the pilot nodes on the m th worker $\Rightarrow \bar{n}_m = l + n_m$.
- Denote the corresponding sub-adjacency matrix as $A^{(\mathcal{S}_m)} \in \mathbb{R}^{\bar{n}_m \times l}$.
- Permute the row indexes of $A^{(\mathcal{S}_m)}$ to ensure that $A^{\left(\mathcal{S}_m\right)} = \left(A_1^{(\mathcal{S}_m)\top}, A_2^{(\mathcal{S}_m)^\top}\right)^\top$ with $A_1^{(\mathcal{S}_m)} = A_0$.
- Let $D_{ii}^{(\mathcal{S}_m)} = \sum_j A_{ij}^{(\mathcal{S}_m)}$ and $F_{jj}^{(\mathcal{S}_m)} = \sum_i A_{ij}^{(\mathcal{S}_m)}$ be the out- and in-degrees of node i and j in the subnetwork on worker m.
- Define

$$D^{(\mathcal{S}_m)} = \mathrm{diag}\left\{D_{ii}^{(\mathcal{S}_m)} : 1 \le i \le \bar{n}_m\right\} \in \mathbb{R}^{\bar{n}_m \times \bar{n}_m}$$

$$F^{(\mathcal{S}_m)} = \mathrm{diag}\left\{F_{jj}^{(\mathcal{S}_m)} : 1 \le j \le l\right\} \in \mathbb{R}^{l \times l}$$

Community Detection on Workers

- The Laplacian version of $A^{(\mathcal{S}_m)}$ is given by

$$L^{(\mathcal{S}_m)} = \left(D^{(\mathcal{S}_m)}\right)^{-1/2} A^{(\mathcal{S}_m)} \left(F^{(\mathcal{S}_m)}\right)^{-1/2} \in \mathbb{R}^{\bar{n}_m \times l}$$

- Perform SVD using $L^{(\mathcal{S}_m)}$ and denote the top K left singular vector matrix as $\widehat{U}^{(\mathcal{S}_m)}$.
- For the i th $(l+1 \leq i \leq \bar{n}_m)$ node in $\mathcal{S}_m$, the cluster label g_i is estimated by

$$\widehat{g}_i = \mathrm{argmin}_{1 \leq k \leq K, i_k \in \mathcal{C}} \left\| \widehat{U}_i^{(\mathcal{S}_m)} - \widehat{U}_{i_k}^{(\mathcal{S}_m)} \right\|_2 .$$

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Extend to Degree-corrected SBM

Let $\Gamma = \text{diag}\{\Gamma_i, 1 \le i \le N\} \in \mathbb{R}^{N\times N} \Rightarrow \mathbb{E}(A) = \Gamma\Theta B\Theta^\top\Gamma$

Algorithm 4: Regularized Distributed Community Detection (r-DCD)

Input: Adjacency matrix A_0; sub-adjacency matrices $\{A^{(\mathcal{S}_m)}\}_{m=1,\cdots,M}$; regularization parameter τ; number of communities K.

Output: Membership matrix $\widehat{\Theta}$

STEP 1 PILOT-BASED NETWORK SPECTRAL CLUSTERING ON MASTER SERVER

- STEP 1.1 Let $L_{0\tau} = D_{0\tau}^{-1/2} A_0 D_{0\tau}^{-1/2}$, where $D_{0\tau} = D_0 + \tau I$. Conduct eigen-decomposition of $L_{0\tau}$ and extract the top K eigenvectors (denoted in matrix $\widehat{U}_0$).
- STEP 1.2 Normalize each row of $\widehat{U}_0$ with unit L_2-norm and obtain $\widehat{U}_{0\tau}$.
- STEP 1.3 Conduct k-means algorithm on $\widehat{U}_{0\tau}$ and obtain clustering centers $\widehat{C}_0 = \left(\widehat{C}_{0k} : 1 \le k \le K\right)^\top$.

STEP 2 BROADCAST PSEUDO CENTERS TO WORKERS

- STEP 2.1 Determine the indexes of the kth pseudo centers as $i_k = \arg\min_i \|\widehat{U}_{0\tau,i} - \widehat{C}_{0k}\|_2^2$, where $\widehat{U}_{0\tau,i}$ is the ith row vector of $\widehat{U}_{0\tau}$.
- STEP 2.2 Broadcast the index set of pseudo centers $\mathcal{C} = \{i_1, \cdots, i_K\}$ to workers.

STEP 3 COMMUNITY DETECTION ON WORKERS

- STEP 3.1 Let $L_\tau^{(\mathcal{S}_m)} = (D^{(\mathcal{S}_m)} + \tau I)^{-1/2} A^{(\mathcal{S}_m)} (F^{(\mathcal{S}_m)} + \tau I)^{-1/2}$. Perform singular value decomposition using $L_\tau^{(\mathcal{S}_m)}$ and denote the top K left singular vector matrix as $\widehat{U}^{(\mathcal{S}_m)}$.
- STEP 3.2 Normalize each row of $\widehat{U}^{(\mathcal{S}_m)}$ with unit L_2-norm and obtain $\widehat{U}_\tau^{(\mathcal{S}_m)}$.
- STEP 3.3 Use (3) to obtain the estimated community labels.

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Theoretical Properties

Theorem 3.1. (Singular Vector Convergence)

Let $\lambda_{1,m} \geq \lambda_{2,m} \geq \cdots \geq \lambda_{K,m} > 0$ be the top K singular values of $\mathcal{L}^{(\mathcal{S}_m)}$. Define $\delta_m = \min_i \mathcal{D}_{\mathrm{ii}}^{(\mathcal{S}_m)}$. Then for any $\epsilon_m > 0$ and $\delta_m > 3\log(n_m + 2l) + 3\log(4/\epsilon_m)$, with probability at least $1 - \epsilon_m$ it holds

$$\left\| \widehat{U}^{(\mathcal{S}_m)} - U^{(\mathcal{S}_m)} Q^{(\mathcal{S}_m)} \right\|_F \leq \frac{8\sqrt{6}}{\lambda_{K,m}} \sqrt{\frac{K \log\left(4\left(n_m + 2l\right)/\epsilon_m\right)}{\delta_m}},$$

where $Q^{(\mathcal{S}_m)} \in \mathbb{R}^{K \times K}$ is a $K \times K$ orthogonal matrix.

- **Remarks:**
  - The error bound is related to the eigen-gap $\lambda_{K,m}$
  - The upper bound is lower if the minimum out-degree δ_m is higher

Theoretical Properties

Theorem 3.2. (Bound if Mis-clustering Rates)

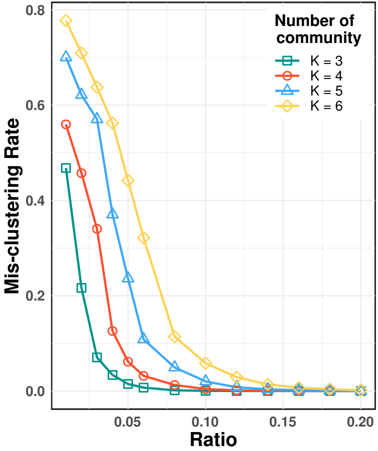
Assume some conditions hold. Let $\mathcal{R}^{(\mathcal{S}_m)}$ denote the ratio of misclustered nodes on worker m, then we have

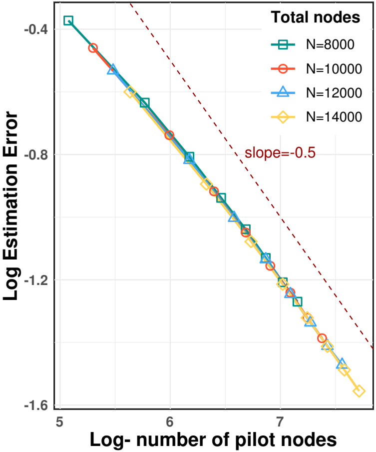
$$\mathcal{R}^{(\mathcal{S}_m)} = O\left(\frac{u_m K^2 \log\left(l/\epsilon_l\right)}{d_0 b_{\min} l \lambda_{K,0}^2} + \frac{K \log\left(4\left(n_m + 2l\right)/\epsilon_m\right)}{\lambda_{K,m}\delta_m} + \frac{u_m \alpha_0^2 K + d_0 \alpha_m^2 K}{d_0 d^2}\right)$$

with probability at least $1 - \epsilon_l - \epsilon_m$, where $u_m = \max_k \pi_k^{(\mathcal{S}_m)}$.

- **Remarks:**
  - The first term is related to the convergence of eigenvectors on the maste
  - The second term is determined by convergence of singular vectors on the mth worker.
  - the third term is mainly related to the unbalanced effect α_m among the workers and α_0 on the master.

Simulation: Pilot Nodes





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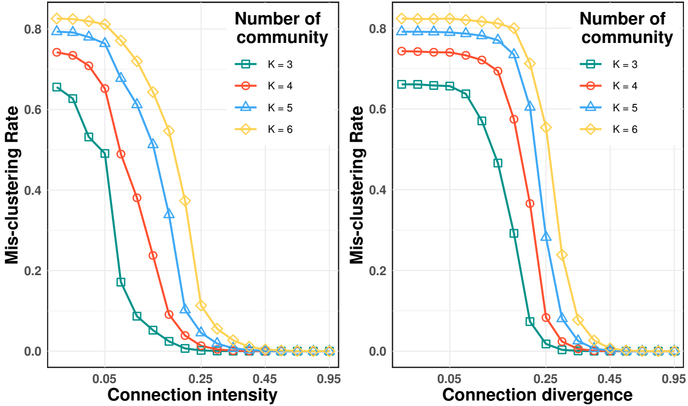
Simulation: Signal Strength

$$B = \nu \left\{ \lambda \boldsymbol{I}_K + (1-\lambda) \mathbf{1}_K \mathbf{1}_K^\top \right\}$$



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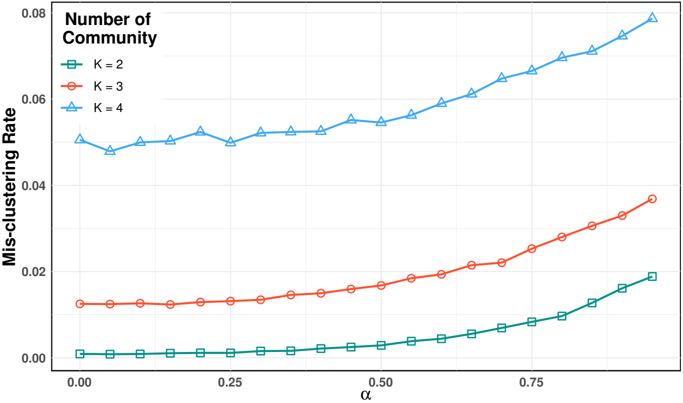
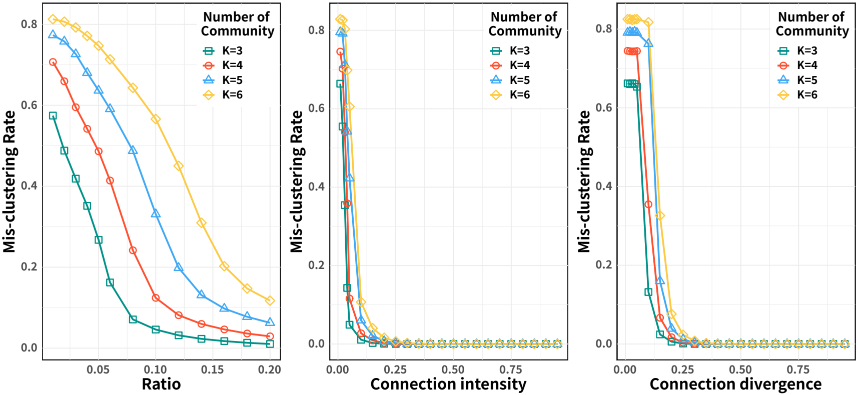
Simulation: Unbalanced Effect

$$\pi_{mk} = \frac{1}{K} + \left(k - \frac{K+1}{2}\right) \operatorname{sign}\left(m - \frac{M+1}{2}\right) \frac{\alpha}{K(K-1)}$$



Simulation: DC-SBM



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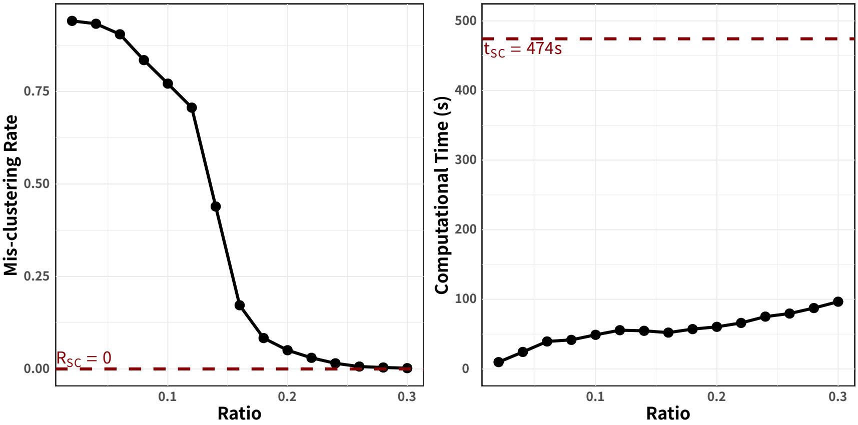
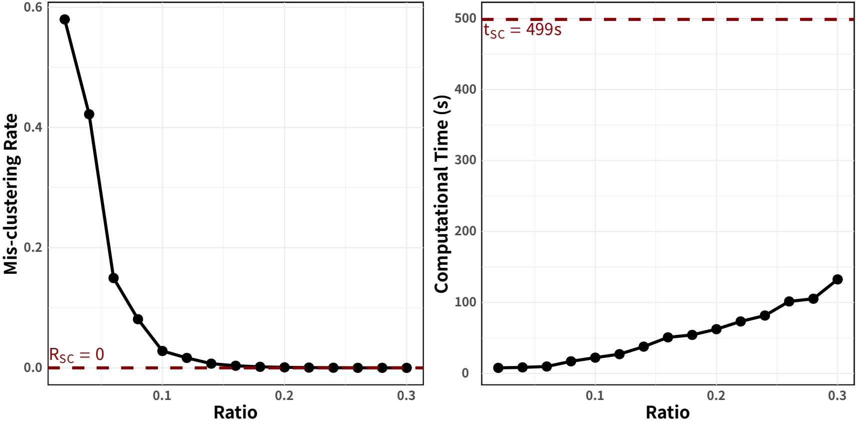
Simulation: Large Scale ($N = 2 \times 10^6$, $K = 20$)



Simulation: Large Scale ($N = 10^7$, $K = 5$)



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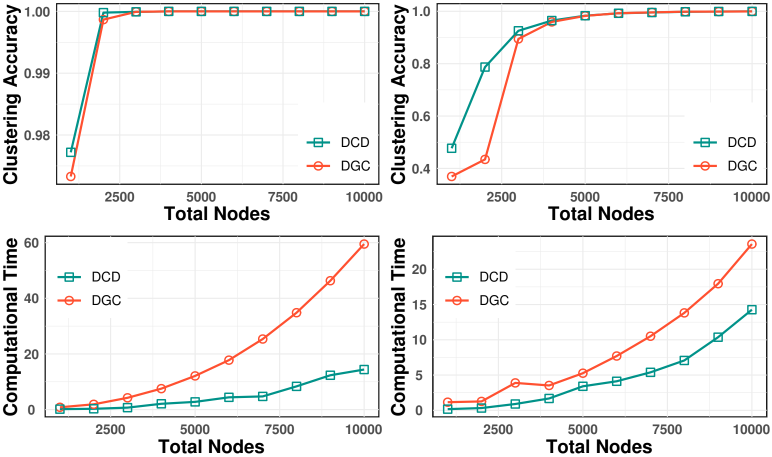
Simulation: Comparison



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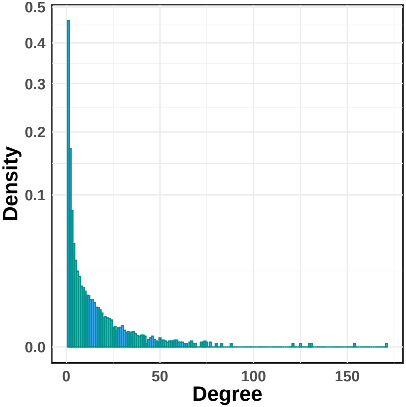
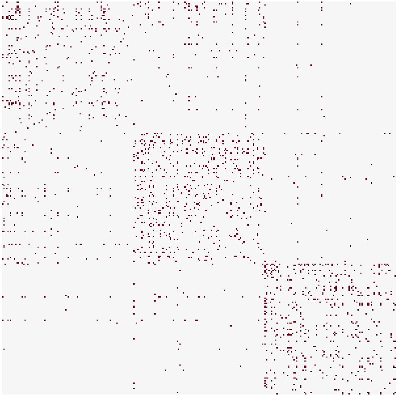
Empirical Study: Pubmed Dataset

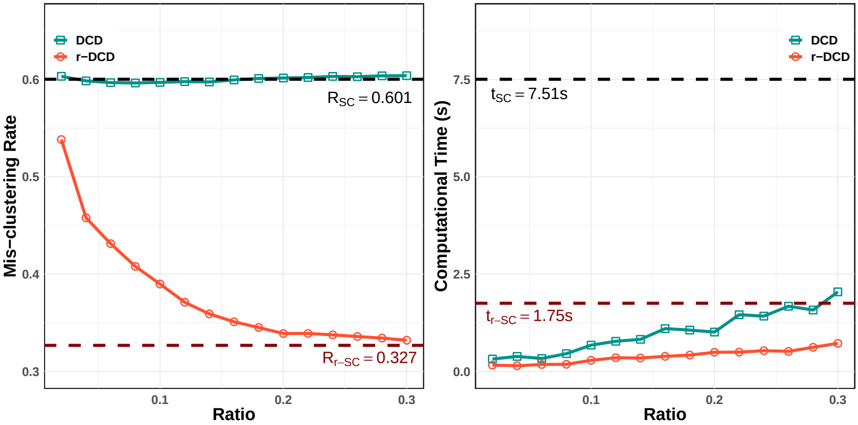
- The Pubmed dataset consists of 19,717 scientific publications
- Each publication is identified as one of the three classes, i.e., Diabetes Mellitus Experimental, Diabetes Mellitus Type 1, Diabetes Mellitus Type 2. $\Rightarrow K = 3$.
- The sizes of the three classes are 4,103, 7,875, and 7,739 respectively.
- The network link is defined using the citation relationships among the publications.
- The resulting network density is 0.028%.

Empirical Study: Pubmed Dataset

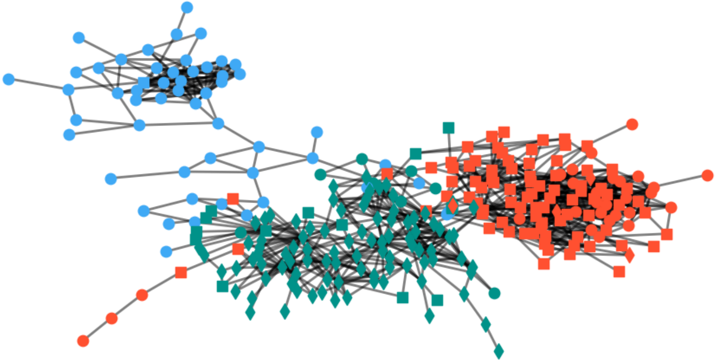




Empirical Study: Pubmed Dataset



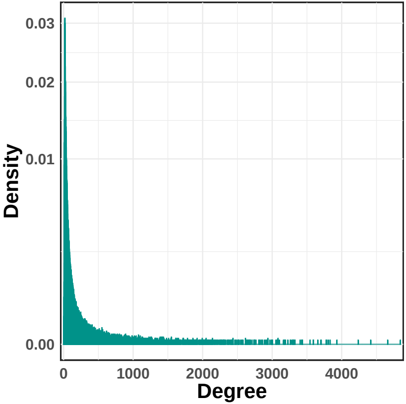
Empirical Study: Pubmed Dataset

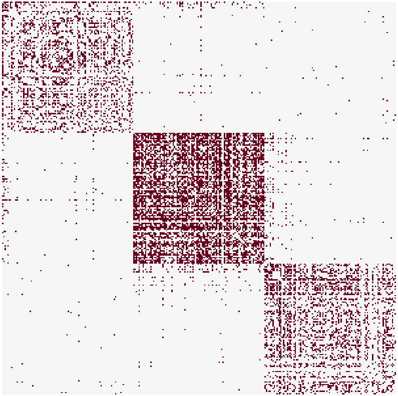


Empirical Study: Pubmed Dataset

- The Yelp is one of the most popular online review platform and the dataset contains 200,193 active users in the network.
- If the ith user is a friend of the jth user, then there is a connection between the two users, i.e., $A_{ij} = 1$
- The resulting network density is 0.031%
- Define the relative density as $\text{RED} = \text{Den}_{\text{between}} \,/\, \text{Den}_{\text{within}}$, where
  - $\text{Den}_{\text{between}} = \sum_{i,j} A_{ij} I(\widehat{g}_i \neq \widehat{g}_j) \,/\, \sum_{i,j} I(\widehat{g}_i \neq \widehat{g}_j)$ is the between-community density
  - $\text{Den}_{\text{within}} = \sum_{i,j} A_{ij} I(\widehat{g}_i = \widehat{g}_j) \,/\, \sum_{i,j} I(\widehat{g}_i = \widehat{g}_j)$ is the within-community density.

Empirical Study: Yelp Dataset





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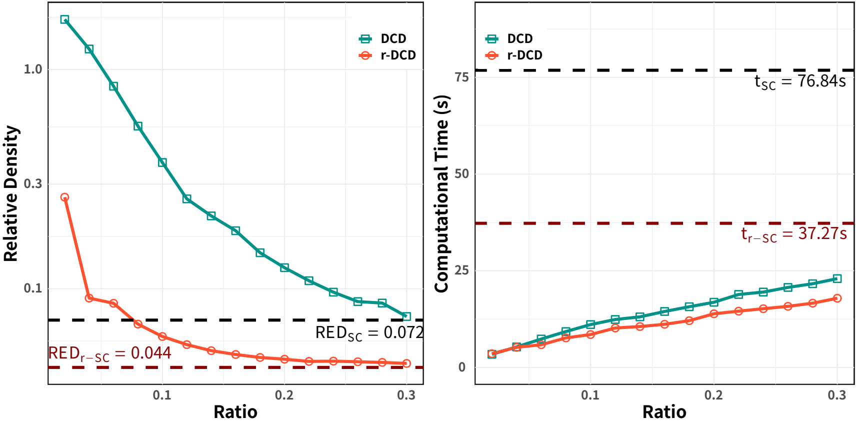
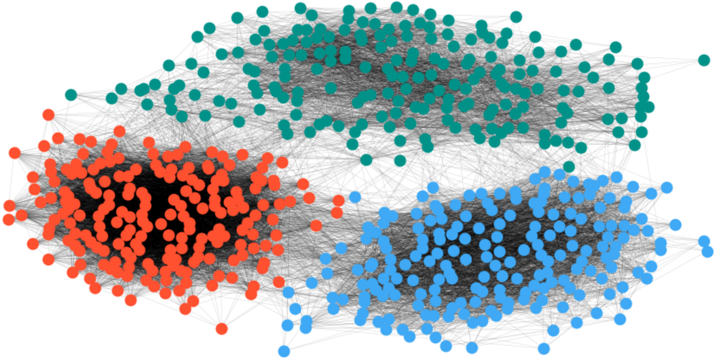
Empirical Study: Yelp Dataset



Empirical Study: Yelp Dataset



Conclusion

- We propose a distributed community detection (DCD) algorithm to tackle community detection task in large scale networks.
  - the communication cost is low
  - no further iterative algorithm is used on workers
  - both the computational complexity and the storage requirements are much lower

- **Paper:** https://www.sciencedirect.com/science/article/pii
- **Code:** https://github.com/Ikerlz/dcd
- **Slide:** https://ikerlz.github.io/uploads/DSBM.pdf

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